Antifield-antibracket formulation of the anti-BRST transformation

1993 J. Phys. A: Math. Gen. 26 6073

(http://iopscience.iop.org/0305-4470/26/21/045)

J. Phys. A: Math. Gen. 26 (1993) 6073–6083. Printed in the UK

Antifield–antibracket formulation of the anti-BRST transformation

Philippe Grégoire† and Marc Henneaux‡
Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C P 231, B-1050 Bruxelles, Belgium

Received 11 January 1993, in final form 13 May 1993

Abstract. The antifield–antibracket implementation of the anti-BRST transformation is given for an arbitrary irreducible gauge invariant action. As in the Hamiltonian case, this is done by duplicating the gauge symmetries and by constructing directly the sum of the BRST and anti-BRST transformations. The gauge fixing process is then derived along the lines of the standard antifield–antibracket formalism. This establishes the equivalence of both formulations in a straightforward way. A brief discussion of the reducible case is reported at the end.

1. Introduction

Recently, the Lagrangian BRST–anti-BRST formalism has attracted considerable attention; different viewpoints have been presented in [1–8]. In accordance with the ideas presented for the Hamiltonian case in [9, 10], we show in this paper (i) that the most convenient way to construct the BRST and anti-BRST transformations is to deal directly with their sum; and (ii) that the proper setting for the whole construction is that of homological perturbation theory [11].

Our starting point is an irreducible gauge invariant action $S_0[\phi]$ where the coordinates $\phi = (\phi^1, \ldots, \phi^n)$ have Grassmann parities $\epsilon(\phi^i) = \epsilon_i$. The gauge generators $R^i_\alpha[\phi]$ with $\alpha = 1, \ldots, N$ are such that

$$\frac{\overleftarrow{\delta}\, S_0}{\delta\phi^i} R^i_\alpha = 0 \qquad \alpha = 1, \ldots, N \tag{1}$$

with $\epsilon(R^i_\alpha) = \epsilon_i + \epsilon_\alpha$. Thus the classical action is invariant under the gauge transformations $\delta_\xi \phi^i = R^i_\alpha[\phi]\xi^\alpha$ with $\epsilon(\xi^\alpha) = \epsilon_\alpha$. We assume that all the gauge generators are independent and form a complete set. If Γ denotes the set of all classical trajectories $\{t \longrightarrow \phi(t)\}$, then the equations of motion

$$G_i \equiv \frac{\overleftarrow{\delta}\, S_0}{\delta\phi^i} = 0 \tag{2}$$

define the stationary surface Σ in Γ. From the completeness assumption about gauge generators, one deduces that

$$\frac{\overleftarrow{\delta}\, R^i_\alpha}{\delta\phi^j} R^j_\beta - (-)^{\epsilon_\alpha\epsilon_\beta} \frac{\overleftarrow{\delta}\, R^i_\beta}{\delta\phi^j} R^j_\alpha \approx -R^i_\gamma C^\gamma_{\alpha\beta} \tag{3}$$

† Chercheur IRSIA.
‡ Also at centro de Estudios Cientificos de Santiago, Casilla 16443, Santiago 9, Chile.

where $\approx$ means that the equality holds on the stationary surface; such an equality is often referred to as a *weak equality*.

2. The antifield–antibracket standard BRST theory

To warm up, let us recall the key ideas of the standard BRST theory in its Lagrangian formulation [12, 13]. The central feature of the antifield–antibracket theory is the construction of the BRST differential s, which captures gauge invariance, and of the BRST invariant extension S of the classical action S_0. The differential s and the BRST invariant action S are constructed at the same time; if $(\cdot, \cdot)$ denotes the antibracket of Batalin and Vilkovisky [12], then S is the unique (up to a canonical transformation) solution of the so-called master equation $(S, S) = 0$ that satisfies the boundary conditions $S = S_0 + \phi_i^* R^i_\alpha \eta^\alpha + \cdots$. We denote by ϕ^A the fields (the original variables ϕ^i and the ghosts η^α) and by ϕ^*_A the antifields (the ϕ^*_i and the η^*_α) so that one has $(\phi^A, \phi^*_B) = \delta^A_B$. The BRST transformation takes the form $s = (\cdot, S)$.

The resolution of the master equation is carried out using homological pertubation theory. It proceeds as follows. The BRST operator contains two crucial differentials

$$s = \delta + D + \cdots \tag{4}$$

playing two distinct roles. The first, denoted δ, is the Koszul–Tate differential whereas the second, denoted D, is a model for the longitudinal exterior differential along the gauge orbits ([13] and references therein). The Koszul–Tate operator δ implements the equations of motion in homology, i.e. it provides a resolution of the algebra of smooth functions defined on the stationnary surface $C^\infty(\Sigma)$, while D takes into account the gauge invariances (on Σ). Within this formalism, the master equation is equivalent to the following family of equations

$$\delta \overset{(k)}{S} = \overset{(k-1)}{D} [\overset{(0)}{S}, \ldots, \overset{(k-1)}{S}] \tag{5}$$

where $S = \sum_{k=0}^{\infty} \overset{(k)}{S}$, $\mathrm{res}(\overset{(k)}{S}) = k$, $gh(S) = 0$ and $\overset{(k-1)}{D}$ has the property of being closed by virtue of the Jacobi identity for the antibracket. The equations (5) are solved recursively. The existence of solutions follows from the acyclicity of the Koszul–Tate operator with a positive degree of resolution.

The solution S of the master equation is the canonical generator of the BRST transformation. Since, s contains both δ and D, the Koszul–Tate differential and the extended longitudinal exterior operator can be expressed through the antibracket. One has

$$D\phi^B = (\phi^B, S)|_{\phi^*_A=0} \tag{6}$$

$$\delta\phi^*_B = (\phi^*_B, S)|_{\eta^\alpha=0}. \tag{7}$$

The construction of a BRST invariant gauge fixed action for the path integral is then performed through the choice of a *gauge-fixing fermion* $\Psi(\phi^A)$ that (traditionally) depends only on the fields ϕ^A:

$$S_\Psi = S\left[\phi^A, \phi^*_A = \frac{\overleftarrow{\delta}\Psi}{\delta\phi^A}\right]. \tag{8}$$

3. The main ideas underlying the BRST–anti-BRST theory

The BRST–anti-BRST algebra is defined by

$$s_1^2 = 0 = s_2^2 \qquad s_1 s_2 + s_2 s_1 = 0 \tag{9}$$

where s_1 and s_2 are, respectively, the BRST and the anti-BRST operators. These operators must be such that their cohomology in degree zero is given by the physical observables. We have shown in [9, 10] that the most expedient way to construct the BRST and anti-BRST generators is to deal directly with the sum

$$s = s_1 + s_2. \tag{10}$$

Because s_1 and s_2 anticommute and are both nilpotent, their sum s is also nilpotent. Conversely, any nilpotent operator that splits into two terms as in (10) implies a BRST–anti-BRST algebra for its separate parts s_1 and s_2. Hence, provided one ensures that s does indeed split as in (10), one can replace (9) by the single equation

$$s^2 = 0. \tag{11}$$

In order to ensure that s splits into just two parts, we found it necessary to introduce a bidegree that distinguishes between s_1 and s_2. That bidegree is called the 'ghost bidegree' and denoted bigh $= (gh_1, gh_2)$ and is defined so that

$$\text{bigh}(s_1) = (1, 0) \qquad \text{and} \qquad \text{bigh}(s_2) = (0, 1). \tag{12}$$

Now, the derivation s is not only nilpotent, but it also possesses the following crucial feature: its action on the original classical fields starts like

$$s\phi^i = R^i_\alpha(\text{ghosts})^\alpha + R^i_\alpha(\text{antighosts})^\alpha + \text{'more'} \tag{13}$$

i.e. the given gauge transformations appear twice in s. They are paired once with the antighosts† and once with the ghosts. One can thus view the (nilpotent) sum s of s_1 and s_2 as the BRST generator corresponding to the redundant description of the gauge symmetries obtained by duplicating the gauge generators. To construct s (and hence, s_1 and s_2), one can accordingly *simply follow the standard BRST procedure for reducible systems*, paying due attention to the bidegrees.

This approach was shown to be direct and effective for the Hamiltonian formalism in [9, 10]. We show below that it also works in the Lagrangian case. There is, however, one complication with respect to the Hamiltonian case. This complication has to do with the antibracket, which has ghost degree 1 in the standard antifield formalism. In order to maintain the symmetry between the ghosts and the antighosts, it will turn out to be necessary to duplicate the antibracket as well and to introduce one antibracket of bidegree $(1, 0)$ and one antibracket of bidegree $(0, 1)$. This problem is dealt with in section 5.

† The antighosts should not be confused with the antifields. Even though they have the same value of $gh_1 - gh_2$, they have different bidegrees and play quite different roles. They appear as non-minimal variables in the standard approach to BRST.

4. The spectrum of ghosts

Let us thus duplicate the gauge generators R^i_α, i.e. let us replace the original form of the gauge transformations $\delta_\xi \phi^i = R^i_\alpha \xi^\alpha$ by the equivalent redundant form $\delta_\xi \phi^i = R^i_a \xi^a$, with $R^i_a \equiv (R^i_\alpha, R^i_\alpha)$. We must introduce the corresponding reducibility functions $Z^a_\alpha \equiv (-\delta^\beta_\alpha, \delta^\beta_\alpha)$. Following the standard rules for BRST construction, one associates with the (new reducible) gauge generators R^i_a and with the reducibility functions Z^a_α the ghosts and the ghosts for ghosts

$$\eta^a \equiv (\eta^\alpha_{(1)}, \eta^\alpha_{(2)}) \qquad \text{and} \qquad \pi^\alpha \tag{14}$$

with $gh(\eta^a) = 1$, $gh(\pi^\alpha) = 2$ and $\epsilon(\eta^\alpha_{(1)}) = \epsilon(\eta^\alpha_{(2)}) = \epsilon_\alpha + 1$, $\epsilon(\pi^\alpha) = \epsilon_\alpha$. As explained above, we define a bidegree that distinguishes between the ghosts $\eta^\alpha_{(1)}$ and the antighosts $\eta^\alpha_{(2)}$ by setting

$$\text{bigh}(\eta^\alpha_{(1)}) = (1, 0) \tag{15}$$

$$\text{bigh}(\eta^\alpha_{(2)}) = (0, 1) \tag{16}$$

$$\text{bigh}(\pi^\alpha) = (1, 1). \tag{17}$$

This motivates the alternative notation $\overset{(1,0)}{\eta^\alpha_{(1)}}, \overset{(0,1)}{\eta^\alpha_{(2)}}, \overset{(1,1)}{\pi^\alpha}$ for $\eta^\alpha_{(1)}, \eta^\alpha_{(2)}$ and π^α, respectively.

Following the standard rules of ordinary BRST theory, one defines on Σ the longitudinal exterior derivative associated with the redundant description of the gauge symmetries as follows [13]

$$D\phi^i = R^i_\alpha (\overset{(1,0)}{\eta^\alpha_{(1)}} + \overset{(0,1)}{\eta^\alpha_{(2)}}) \tag{18}$$

$$D \overset{(1,0)}{\eta^\alpha_{(1)}} = - \overset{(1,1)}{\pi^\alpha} + \tfrac{1}{2}(-)^{\epsilon_\gamma+1} C^\alpha_{\beta\gamma} (\overset{(1,0)}{\eta^\beta_{(1)}} \overset{(1,0)}{\eta^\gamma_{(1)}} + \overset{(1,0)}{\eta^\beta_{(1)}} \overset{(0,1)}{\eta^\gamma_{(2)}}) \tag{19}$$

$$D \overset{(1,0)}{\eta^\alpha_{(1)}} = \overset{(1,1)}{\pi^\alpha} + \tfrac{1}{2}(-)^{\epsilon_\gamma+1} C^\alpha_{\beta\gamma} (\overset{(0,1)}{\eta^\beta_{(2)}} \overset{(0,1)}{\eta^\gamma_{(2)}} + \overset{(0,1)}{\eta^\beta_{(2)}} \overset{(1,0)}{\eta^\gamma_{(1)}}). \tag{20}$$

The action of D on the ghosts for ghosts $\overset{(1,1)}{\pi^\alpha}$ is chosen in such a way that $D^2 \approx 0$. Due to the bigraduation of the polynomial algebra

$$K^{*,*} = C[\overset{(1,0)}{\eta^\alpha_{(1)}}, \overset{(0,1)}{\eta^\alpha_{(2)}}, \overset{(1,1)}{\pi^\alpha}] \otimes C^\infty(\Gamma)$$

the total differential D splits as $D = D_1 + D_2$, with $\text{bigh}(D_1) = (1, 0)$ and $\text{bigh}(D_2) = (0, 1)$:

$$D_1\phi^i = R^i_\alpha \overset{(1,0)}{\eta^\alpha_{(1)}} \qquad\qquad D_2\phi^i = R^i_\alpha \overset{(0,1)}{\eta^\alpha_{(2)}} \tag{21}$$

$$D_1 \overset{(1,0)}{\eta^\alpha_{(1)}} = \tfrac{1}{2}(-)^{\epsilon_\gamma+1} C^\alpha_{\beta\gamma} \overset{(1,0)}{\eta^\beta_{(1)}} \overset{(1,0)}{\eta^\gamma_{(1)}} \tag{22}$$

$$D_2 \overset{(1,0)}{\eta^\alpha_{(1)}} = - \overset{(1,1)}{\pi^\alpha} + \tfrac{1}{2}(-)^{\epsilon_\gamma+1} C^\alpha_{\beta\gamma} \overset{(1,0)}{\eta^\beta_{(1)}} \overset{(0,1)}{\eta^\gamma_{(2)}} \tag{23}$$

$$D_1 \overset{(0,1)}{\eta^\alpha_{(2)}} = \overset{(1,1)}{\pi^\alpha} + \tfrac{1}{2}(-)^{\epsilon_\gamma+1} C^\alpha_{\beta\gamma} \overset{(0,1)}{\eta^\beta_{(2)}} \overset{(1,0)}{\eta^\gamma_{(1)}} \tag{24}$$

$$D_2 \overset{(0,1)}{\eta^\alpha_{(2)}} = \tfrac{1}{2}(-)^{\epsilon_\gamma+1} C^\alpha_{\beta\gamma} \overset{(0,1)}{\eta^\beta_{(2)}} \overset{(0,1)}{\eta^\gamma_{(2)}}. \tag{25}$$

The weak nilpotency of D implies $D_1^2 \approx D_2^2 \approx 0$ and $D_1 D_2 + D_2 D_1 \approx 0$.

5. The antibracket structure

In the standard antifield formalism, there is one antibracket which has ghost number 1. In order to make this structure compatible with the bidegree [14–16] while simultaneously preserving the symmetry between the two degrees gh_1 and gh_2, it is necessary to introduce here two antibrackets, one with bidegree $(1, 0)$, the other with bidegree $(0, 1)$. There is no such difficulty in the Hamiltonian formalism because the Poisson bracket has ghost number zero. For each field $\phi^A \equiv (\phi^i, \eta^\alpha_{(1)}, \eta^\alpha_{(2)}, \pi^\alpha)$, one thus introduces two antifields $\phi^{*(1)}_A$ and $\phi^{*(2)}_A$, *one conjugate to* ϕ^A *in the first antibracket, the other conjugate to* ϕ^A *in the second.* Thus, we have

$$\begin{array}{lllll} \overset{(-1,0)}{\phi^{*(1)}_i} & \overset{(-2,0)}{\eta^{*(11)}_\alpha} & \overset{(-1,-1)}{\eta^{*(12)}_\alpha} & \overset{(-2,-1)}{\pi^{*(1)}_\alpha} & (\equiv \phi^{*(1)}_A) \\ \overset{(0,-1)}{\phi^{*(2)}_i} & \overset{(-1,-1)}{\eta^{*(21)}_\alpha} & \overset{(0,-2)}{\eta^{*(22)}_\alpha} & \overset{\pi}{0}{}^{*(-2)}_{-1}\,\alpha 2(-1,-2) & (\equiv \phi^{*(2)}_A) \end{array} \tag{26}$$

where the superscript (a, b) denotes the ghost bidegree.

In analogy with the standard BRST formalism (equation (6)), we require that $D\phi^A$ be generated through the antibracket $(\cdot,\cdot) = (\cdot,\cdot)_1 + (\cdot,\cdot)_2$ with a generator S of bidegree $(0, 0)$. This implies that $D_1\phi^A$ and $D_2\phi^A$ are generated through $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1$, respectively, with the same generator

$$D_1\phi^A = (\phi^A, S)_1|_{\text{antifields}=0} \qquad \text{and} \qquad D_2\phi^A = (\phi^A, S)_2|_{\text{antifields}=0}. \tag{27}$$

It is easy to check that S must start as follows,

$$\begin{aligned} S &= S_0 + \overset{(-1,0)}{\phi^{*(1)}_i} R^i_\alpha \overset{(1,0)}{\eta^\alpha_{(1)}} + \overset{(0,-1)}{\phi^{*(2)}_i} R^i_\alpha \overset{(0,1)}{\eta^\alpha_{(2)}} + (\overset{(-1,-1)}{\eta^{*(12)}_\alpha} - \overset{(-1,-1)}{\eta^{*(21)}_\alpha}) \overset{(1,1)}{\pi^\alpha} + \cdots \\ &= \overset{(0)}{S} + \overset{(1)}{S} + \overset{(2)}{S} + \cdots. \end{aligned} \tag{28}$$

The first term $S_0 \equiv \overset{(0)}{S}$ is, in fact, not dictated by (21)–(25), but is included for the description of the Koszul–Tate biresolution, to which we now turn.

6. The Koszul–Tate biresolution

Following (7), we define the Koszul–Tate differential δ by taking the antibracket of the antifields with S and then setting the ghosts equal to zero. One gets from (28)

$$\delta = \delta_1 + \delta_2 \tag{29}$$

with

$$\delta_1 \overset{(-1,0)}{q^{*(1)}_i} = (\overset{(-1,0)}{\phi^{*(1)}_i}, \overset{(0)}{S})_1 = -\frac{\vec{\delta}\, S_0}{\delta\phi^i} \tag{30}$$

$$\delta_2 \overset{(0,-1)}{q^{*(2)}_i} = (\overset{(0,-1)}{\phi^{*(2)}_i}, \overset{(0)}{S})_2 = -\frac{\vec{\delta}\, S_0}{\delta\phi^i} \tag{31}$$

$$\delta_1 \overset{(0,-1)}{\phi^{*(2)}_i} = 0 = \delta_2 \overset{(1,0)}{\phi^{*(1)}_i} \tag{32}$$

and

$$\delta_1 \overset{(-2,0)}{\eta^{*(11)}_\alpha} = (\overset{(-2,0)}{\eta^{*(11)}_\alpha}, \overset{(1)}{S})_1 = -(-)^{\epsilon_\alpha} \overset{(-1,0)}{\phi^{*(1)}_i} R^i_\alpha \tag{33}$$

$$\delta_1 \overset{(-1,-1)}{\eta^{*(21)}_\alpha} = (\overset{(-1,-1)}{\eta^{*(21)}_\alpha}, \overset{(1)}{S})_1 = -(-)^{\epsilon_\alpha} \overset{(-1,0)}{\phi^{*(2)}_i} R^i_\alpha \tag{34}$$

$$\delta_1 \overset{(-1,-1)}{\eta^{*(12)}_\alpha} = (\overset{(-1,-1)}{\eta^{*(12)}_\alpha}, \overset{(1)}{S})_2 = -(-)^{\epsilon_\alpha} \overset{(-1,0)}{\phi^{*(1)}_i} R^i_\alpha \tag{35}$$

$$\delta_1 \overset{(0,-2)}{\eta^{*(22)}_\alpha} = (\overset{(0,-2)}{\eta^{*(22)}_\alpha}, \overset{(1)}{S})_1 = -(-)^{\epsilon_\alpha} \overset{(-1,0)}{\phi^{*(2)}_i} R^i_\alpha \tag{36}$$

$$\delta_2 \overset{(-2,0)}{\eta^{*(11)}_\alpha} = (\overset{(-2,0)}{\eta^{*(11)}_\alpha}, \overset{(1)}{S})_2 = 0 \tag{37}$$

$$\delta_2 \overset{(-1,-1)}{\eta^{*(21)}_\alpha} = (\overset{(-1,-1)}{\eta^{*(21)}_\alpha}, \overset{(1)}{S})_2 = 0 \tag{38}$$

$$\delta_1 \overset{(-1,-1)}{\eta^{*(12)}_\alpha} = (\overset{(-1,-1)}{\eta^{*(12)}_\alpha}, \overset{(1)}{S})_1 = 0 \tag{39}$$

$$\delta_1 \overset{(0,-2)}{\eta^{*(22)}_\alpha} = (\overset{(0,-2)}{\eta^{*(22)}_\alpha}, \overset{(1)}{S})_1 = 0. \tag{40}$$

One also finds $\delta_1\pi^{*(2)}_\alpha = 0 = \delta_2\pi^{*(1)}_\alpha$ and $\delta_2\pi^{*(2)}_\alpha = \eta^{*(12)}_\alpha - \eta^{*(21)}_\alpha = \delta_1\pi^{*(2)}_\alpha$. However, as we shall see presently, this last relation must be modified.

Indeed, there are two problems with this definition of the Koszul–Tate complex. These problems originate from the fact that we have departed from the usual antifield formalism by duplicating the antibracket and are:

(i) the operator δ fails to be nilpotent on π^*; and

(ii) there are non-trivial cycles with a positive degree of resolution. For instance, $\overset{(0,-1)}{\phi^{*(2)}_i} - \overset{(-1,0)}{\phi^{*(1)}_i}$, $\overset{(0,-1)}{\phi^{*(2)}_i}$ and $\overset{(-1,0)}{\phi^{*(1)}_i}$ are δ, δ_1 and δ_2-cycles, respectively. If the antifields $\phi^{*(1)}_A$ and $\phi^{*(2)}_A$ were equal, none of these problems would arise. This suggests solving the difficulty by introducing further variables that kill the difference $\phi^{*(1)}_A - \phi^{*(2)}_A$ in homology. These extra variables are called the 'bar variables' and denoted by $\bar{\phi}_A$. So we set

$$\text{bigh}\,\bar{\phi}_A = (-gh_1\phi^A - 1, -gh_2\phi^A - 1) \tag{41}$$

and define

$$V\bar{\phi}_A = \phi^{*(2)}_A - \phi^{*(1)}_A. \tag{42}$$

The operator V is the same as the one introduced in [2], and has been given a geometrical interpretation in [15]. It is nilpotent and it acts as a derivation for the antibracket (,). Only the sum of the antifields (say) survives in its homology. Since $\delta^2\pi^*$ fails to be zero by terms that are in the homology of V, we can cure the definition of $\delta\pi^*$ by adding terms proportional to the bar variables

$$\begin{aligned}
\delta_1 \overset{(-2,-1)}{\pi^{*(1)}_\alpha} = \delta_2 \overset{(-1,-2)}{\pi^{*(2)}_\alpha} &= (\overset{(-2,-1)}{\pi^{*(1)}_\alpha}, (\overset{(-1,-1)}{\eta^{*(12)}_\beta} - \overset{(-1,-1)}{\eta^{*(21)}_\beta} + \overset{(-1-1)}{\bar{\phi}_i} R^i_\beta)\overset{(1,1)}{\pi^\beta})_1 \\
&= (\overset{(-1,-2)}{\pi^{*(2)}_\alpha} (\overset{(-1,-1)}{\eta^{*(12)}_\beta} - \overset{(-1,-1)}{\eta^{*(21)}_\beta} + \overset{(-1,-1)}{\bar{\phi}_i} R^i_\beta)\overset{(1,1)}{\pi^\beta})_2 \\
&= \overset{(-1,-1)}{\eta^{*(12)}_\alpha} - \overset{(-1,-1)}{\eta^{*(21)}_\alpha} + \overset{(-1,-1)}{\bar{q}_i} R^i_\alpha.
\end{aligned} \tag{43}$$

This amounts to adding to $\overset{(2)}{S}$ the term $\overset{(-1,-1)}{\bar{\phi}_i} R^i_\alpha \overset{(1,1)}{\pi^\beta}$, i.e. to modifying S as

$$S = S_0 + \overset{(-1,0)}{\phi_i^{*(1)}} R^i_\alpha \overset{(1,0)}{\eta^\alpha_{(1)}} + \overset{(0,-1)}{\phi_i^{*(2)}} R^i_\alpha \overset{(0,1)}{\eta^\alpha_{(2)}} + (\overset{(-1,-1)}{\eta^{*(12)}_\alpha} - \overset{(-1,-1)}{\eta^{*(21)}_\alpha} + \overset{(-1,-1)}{\bar{\phi}_i} R^i_\alpha) \overset{(1,1)}{\pi^\alpha} + \cdots . \tag{44}$$

The complete Koszul–Tate operator δ is the sum of a *canonical* and a *non-canonical* part,

$$\delta = \delta_{\text{canonical}} + V \tag{45}$$

with V given by (42) and $\delta_{\text{canonical}}$ equal to

$$\delta_{\text{canonical}} = (\cdot, S)|_{\eta=0=\pi} . \tag{46}$$

Because we have included S_0 in S, i.e. $\delta S_0/\delta\phi^i$ in $\delta\phi_i^*$, it is now easy to see that on the polynomial algebra $K_{*,*} = C[\phi_A^{*(1)}, \phi_A^{*(2)}, \bar{\phi}_A] \otimes C^\infty(\Gamma)$ bigraded by the resolution bidegree, one has

(i) $\text{bires}(\delta_1) = (-1, 0)$, $\text{bires}(\delta_2) = (0, -1)$.

(ii) $H_{0,0}(\delta_1) = H_{0,0}(\delta_2) = H_0(\delta) = C^\infty(\Sigma)$, while the other homology spaces are trivial.

In other words, the bicomplex $(K_{*,*}, \delta = \delta_1 + \delta_2)$ is a biresolution of the algebra $C^\infty(\Sigma)$ in the sense of [10].

7. The master equation

The total BRST transformation s is equal to $\delta + D + \cdots$, where the additional terms are chosen in such a way that $s^2 = 0$. From the previous sections, we should not expect S to be canonically generated. Rather, we are led to try to complete S in such a way that

$$s = (\cdot, S) + V. \tag{47}$$

The requirement that s be nilpotent is equivalent to the 'master equation' for BRST–anti-BRST theory,

$$\tfrac{1}{2}(S, S) + VS = 0. \tag{48}$$

It is easy to see that equation (48), in turn, is equivalent to the family of equations

$$\delta \overset{(k)}{S} = \overset{(k-1)}{D} [\overset{(0)}{S}, \ldots, \overset{(k-1)}{S}] \qquad k \geqslant 1 \tag{49}$$

where $S = \sum_{k=0}^{\infty} \overset{(k)}{S}$ and $\text{res}(\overset{(k)}{S}) = k$. These equations take exactly the same form as in ordinary BRST theory and can be solved in exactly the same recursive method of homological perturbation theory. Indeed, $\overset{(k-1)}{D}$ is δ-closed for $k > 1$ by the Jacobi identity. Because δ is acyclic at higher resolution degree, there exits, for any $k \geqslant 1$, a solution of (49) compatible with the boundary conditions. The case, $k = 1$, is easily solved and it is straightforward to check that $\overset{(1)}{S}$ is purely a boundary term.

Furthermore, the general formalism controlling the bidegree developed in [10] applies and guarantees that the solution S not only exists but can also be taken to be of bidegree $(0, 0)$ (see the so-called 'positivity theorem' in [10]). Then, S splits as in (10), with

$$s_1 = \tfrac{1}{2}(\cdot, S)_1 + V_1 \tag{50}$$

$$s_2 = \tfrac{1}{2}(\cdot, S)_2 + V_2. \tag{51}$$

while the master equation itself splits as

$$\tfrac{1}{2}(S, S)_1 + V_1 S = 0 = \tfrac{1}{2}(S, S)_2 + V_2 S. \tag{52}$$

Those equations as well as the boundary conditions on S are the same as those appearing in [2] in the context of the $sp(2)$ formalism, but we have focused here on just the BRST–anti-BRST algebra rather than on the full $sp(2)$ algebra. We have thus established that one can define the BRST–anti-BRST algebra for any gauge theory (prior to gauge fixing), as well as equivalence with the $sp(2)$-formalism. Note that S is neither BRST-invariant nor anti-BRST invariant. This is due to the fact that it is not the canonical generator of these transformations. Finally, note also that spacetime locality of S can be proven along the same lines as in the ordinary BRST formalism [13, 17], since the equations (49) for the structure functions take the same form.

8. The gauge-fixing process

A method for gauge fixing S in a manner that preserves both BRST and anti-BRST invariance has been given in [2]. The equivalence of that method with the usual BRST gauge-fixing method is, however, not obvious. Even though equivalence proofs have appeared since then [14, 15], it is of interest to provide a more direct proof. This is done here.

Let us introduce an extra field $\mu^A_{(1)}$ conjugate to $\bar{\phi}_A$ in the first antibracket and let us forget for a moment about the second antibracket:

$$(\mu^A_{(1)}, \bar{\phi}_B)_1 = \delta^A_B. \tag{53}$$

Let us also introduce a field $\rho^A_{(1)}$ conjugate to $\phi^{*(2)}_A$ in the first antibracket, so that all the variables appear in conjugate pairs,

$$(\phi^A, \phi^{*(1)}_B)_1 = \delta^A_B \tag{54}$$

$$(\mu^A_{(1)}, \bar{\phi}_B)_1 = \delta^A_B \tag{55}$$

$$(\phi^{*(2)}_A, \rho^B_{(1)})_1 = \delta^B_A. \tag{56}$$

In the following we shall regard $\phi^{*(2)}_A$ as a field and $\rho^A_{(1)}$ as an antifield.

If one sets $S_1 = S + \phi^{*(2)}_A \mu^A_{(1)}$, then the first equation of (52) is equivalent to the equation

$$(S_1, S_1)_1 = 0. \tag{57}$$

Equation (57) shows that S_1 is s_1-invariant, even though S is not. The additional terms in S_1 restores the canonical structure for the first antibracket with the consequence that

s_1 is now canonically generated with canonical generator $S_1 : s_1 = (\cdot, S_1)_1$. Let $\sigma[\phi^i, \eta^\alpha_{(1)}, q_i^{*(1)}, \eta_\alpha^{*(11)}]$ be the standard minimal solution of the usual master equation $(\sigma,\sigma)_1 = 0$, with the usual boundary conditions. Then, the action $S_1'[\phi^A, \phi_A^{*(a)}] = \sigma - \eta_\alpha^{*(21)}\pi^\alpha + \phi_A^{*(2)}\mu^A_{(1)}$ is a non-minimal solution of $(S_1', S_1')_1 = 0$. Using the standard uniqueness theorems of antifield–antibracket theory, one sees that S_1 is a non-minimal solution of the master equation $(S_1, S_1)_1 = 0$ which can be obtained from S_1' via a canonical transformation. We now apply the standard procedure to fix the gauge recalled in section 2; the gauge fixed action is given by

$$S_\Psi = S_1\left[\phi^A, \mu^A_{(1)}, \phi_A^{*(2)}, \phi_A^{*(1)} = \frac{\overleftarrow{\delta}\Psi}{\delta\phi^A}, \bar{\phi}_A = \frac{\overleftarrow{\delta}\Psi}{\delta\mu^A_{(1)}}, \rho^A_{(1)} = \frac{\overleftarrow{\delta}\Psi}{\delta\phi_A^{*(2)}}\right]. \qquad (58)$$

Let $F[\phi^A]$ be a bosonic functional depending only on ϕ^A, and choose

$$\Psi = \mu^A_{(1)}\frac{\overleftarrow{\delta} F}{\delta\phi^A}. \qquad (59)$$

Then, one has

$$\bar{\phi}_A = \frac{\overleftarrow{\delta} F}{\delta\phi^A} \qquad \text{and} \qquad \phi_A^{*(1)} = \mu^B_{(1)}\frac{\overleftarrow{\delta} F}{\delta\phi^A\delta\phi^B}. \qquad (60)$$

If we introduce Lagrange multipliers $\mu^A_{(2)}$ and λ^A for the gauge choice (60), the usual path integral

$$Z_\Psi = \int \mathcal{D}\phi^A\mathcal{D}\mu^A_{(1)}\mathcal{D}\phi_A^{*(2)}\exp(\mathrm{i}S_\Psi) \qquad (61)$$

with the choice of gauge fermion (59) can be rewritten

$$Z_\Psi = \int \mathcal{D}\phi^A\mathcal{D}\mu^A_{(1)}\mathcal{D}\phi_A^{*(1)}\mathcal{D}\mu^A_{(2)}\mathcal{D}\phi_A^{*(2)}\mathcal{D}\lambda^A\mathcal{D}\bar{\phi}_A\exp(\mathrm{i}S_{\mathrm{eff}}) \qquad (62)$$

where the effective action reads

$$S_{\mathrm{eff}} = S_1 - \phi_A^{*(1)}\mu^A_{(2)} + \mu^A_{(1)}\frac{\overleftarrow{\delta} F}{\delta\phi^B\delta\phi^A}\mu^B_{(2)} + (\bar{\phi}_A - \frac{\overleftarrow{\delta} F}{\delta\phi^A})\lambda^A. \qquad (63)$$

Integrating over the multipliers $\mu^A_{(2)}$ and λ^A forces the choice (60) and yields back the action (58).

The path integral (62) is exactly the one obtained by Batalin *et al* in [2]. However, (62) has been obtained here within the standard antifield–antibracket formalism, and so, equivalence with that formalism is manifest (the choice of non-minimal sector does not modify the physical amplitudes). In particular, the standard Batalin–Vilkovisky theorem shows that (62) is independent of the choice of bosonic functional F. Also, the effective action is guaranteed to be BRST invariant.

One can similarly derive (62) by using the anti-BRST transformation and the second antibracket, since the final result is completely symmetric between ϕ^*_{A1} and ϕ^*_{A2}. One can thus conclude that (62) is not only BRST-invariant but also anti-BRST invariant.

The invariance of the effective action (63) can be checked directly. Indeed, the variation of the effective action (63) under the following 'gauge-fixed' BRST and anti-BRST transformations

$$s_a \phi^A = \epsilon^{ab} \mu_b^A \tag{64}$$

$$s_a \phi_A^{*(b)} = \delta_b^a \frac{\overleftarrow{\delta} S}{\delta \phi^A} (-)^{\epsilon_A+1} \tag{65}$$

$$s_a \bar{\phi}_A = \epsilon^{ab} \phi_A^{*(b)} \tag{66}$$

$$s_a \mu_b^A = \delta_{ab} \lambda^A (-)^{\epsilon_A+1} \tag{67}$$

$$s_a \lambda^A = 0 \tag{68}$$

(where $a, b = 1, 2$ and $\epsilon^{12} = 1$) is easily seen to vanish. To compare (64)–(68) with the standard form of the BRST transformation, we note that in S_{eff}, the fields λ^A, $\bar{\phi}_A$, $\mu_{(2)}^A$ and $\phi^{*(1)}$ can be viewed as auxiliary fields. The symmetry s_1 can be first expressed on ϕ^A, $\phi_A^{*(2)}$ and $\mu_{(1)}^A$ through the usual rule of the antifield–antibracket formalism, that is, $s_1 = (\cdot, S_1)_1$. Then the BRST symmetry can be extended to the auxiliary fields as explained, for instance, in [13]. This yields the above equations. To derive the anti-BRST symmetry, one proceeds similarly and now treats λ^A, $\bar{\phi}_A$, $\mu_{(1)}^A$ and $\phi^{*(2)}$ as auxiliary fields . Note that the symmetries (64)–(68) are defined only modulo skew symmetric combinations of the equations of motion and also that they slightly differ from those presented in [2] where there is a misprint in the transformation of $\mu_{(b)}^A$.

9. The reducible case

The reducible case is treated along the same lines as the irreducible one. The spectrum of ghost is obtained by duplicating the gauge symmetries and the reducibility relations; this duplication is similar to the one considered in the Hamiltonian BRST–anti-BRST formalism (see [16]). The construction of the Koszul–Tate biresolution is performed by requiring (i) the existence of a canonical bistructure and (ii) the acyclicity at a higher degree of resolution. Finally, the gauge-fixing process is absolutely similar to the one presented above.

10. Conclusion

We have shown that the BRST–anti-BRST symmetry can be constructed in the antifield–antibracket formalism by adapting the methods of homological perturbation theory. The cornerstone of our construction relies on the existence of a biresolution of the algebra of smooth functions defined on the stationary surface that allows us to use the positivity theorem. We have also shown that the gauge-fixing process of the BRST–anti-BRST antifield–antibracket formalism can be seen as a particular choice of gauge-fixing fermion in the standard antifield–antibracket formalism, leading to a proof of the equivalence between the BRST–anti-BRST theory and the standard BRST theory in its Lagrangian formulation. Our results also coincide with the $sp(2)$ formalism developed by Batalin *et al* in [2].

Finally, in the case when the gauge algebra closes off-shell, one can take the solution S of the master equation to be linear in the antifields,

$$S = S_0[\phi^i] + \phi_A^{*(a)} s_a \phi^A + \bar{\phi}_A s_1 s_2 \phi^A \tag{69}$$

where $s_a\phi^A$ depends only on the fields. The integration over $\mu^A_{(a)}$, $\phi_A^{*(a)}$ and λ^A is then immediate and yields the familiar path integral [18, 19]

$$Z_F = \int \mathcal{D}\phi^A \exp(\mathrm{i}S_0[\phi^i] + s_1 s_2 F). \tag{70}$$

When the gauge algebra closes on-shell, however, it is not possible to eliminate the extra variables [20] in general.

Acknowledgment

This work has been supported in part by a research contract with the Commission of the European Communities.

References

[1] Spiridonov V P 1988 *Nucl. Phys.* B **308** 257
[2] Batalin I A, Lavrov P M and Tyutin I V 1990 *J. Math. Phys.* **31** 1487
[3] Batalin I A, Lavrov P M and Tyutin I V 1991 *J. Math. Phys.* **32** 532
[4] Gomis J and Roca J 1990 *Nucl. Phys.* B **343** 152
[5] Dur E and Gates S J Jr. 1990 *Nucl. Phys.* B **343** 622
[6] Hull C M, Spence B and Vasquez-Bello J L 1991 *Nucl. Phys.* B **348** 108
[7] Hull C M 1990 *Mod. Phys. Lett.* A **5** 1871
[8] Perry M J and Teo E 1993 *Nucl. Phys.* B **392** 369
[9] Grégoire P and Henneaux M 1992 *Phys. Lett.* B **277** 459
[10] Grégoire P and Henneaux M 1992 *Commun. Math. Phys.* in press (*Preprint* ULB-PMIF-92/06)
[11] Gugenheim V K A M and Stasheff J D 1896 *Bull. Soc. Math. Belg.* **38** 237 and references therein
[12] Batalin I A and Vilkovisky G A 1981 *Phys. Lett.* **102B** 27
[13] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[14] Henneaux M 1991 *Commun. Math. Phys.* **140** 1
[15] Batalin I A, Lavrov P M and Tyutin I V 1991 *J. Math. Phys.* **32** 2513
[16] Grégoire P 1993 La transformation BRST–anti-BRST *PhD Thesis* Brussels
[17] Henneaux M 1992 *Phys. Lett.* **282B** 372
[18] Alvarez-Gaumé L and Baulieu L 1992 *Nucl. Phys.* B **212**
Ore F R and van Nieuwenhuizen P 1982 *Nucl. Phys.* B **204** 317
[19] Baulieu L 1985 *Phys. Rep.* **129** 1
[20] Barnich G, Constantinescu R and Grégoire P 1992 *Phys. Lett.* **293B** 353